Exam Special Relativity
December 11, 2014
Start: 14:00h End: 16:00h

INSTRUCTIONS: This is a closed-book and closed-notes exam. The exam duration is 2 hours. You are allowed to use a numerical (non programmable) calculator. There is a total of 9 points that you can collect. Problems are designed, as much as it is possible, so that you can answer a given part of the problem without necessarily answer other parts. Work by default with SR units.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also loose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need. Write clearly, and draw clearly the spacetime diagrams making use of the squared paper. **Solve the problems on separate sheets: 1 sheet has 4 sides**, e.g. problem 1 on sheet 1 and problem 2 on sheet 2.

USEFUL FORMULAS AND CONSTANTS

$3.00 \times 10^8\,\mathrm{m} = 1\mathrm{s}$ $\qquad (1+x)^n \sim 1 + nx \quad x \ll 1$

$$P_x = \frac{m v_x}{\sqrt{1-v^2}} \qquad E = \frac{m}{\sqrt{1-v^2}}$$

Lorentz transformation from the Other Frame (t', x', y', z') to the Home Frame (t, x, y, z)

$$\Delta t = \gamma \Delta t' + \gamma\beta\Delta x' \qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta x = \gamma \Delta x' + \gamma\beta\Delta t'$$
$$\Delta y = \Delta y'$$
$$\Delta z = \Delta z'$$

$$L_R = \gamma L$$

1. (6 points total) Susan is driving a small spacecraft around an essentially circular orbit at an essentially constant speed of $v = 3/5$. Brian, who is sitting at a fixed position in a space station on the edge of the orbit, measures 200 ns with his high-precision watch when Susan passes by his position the first time (call it event E) and measures 700 ns when Susan passes his position again (call it event F); Brian thus finds that Susan takes 500 ns to complete one full orbit. Susan also measures the time for one orbit on her high-precision watch. This situation is also observed by Alice and Dave, who are passengers on a spacecraft that passes very close to Brian. Alice sits precisely at the front end-point of the spacecraft and happens to be passing Brian just as Susan passes Brian the first time, while Dave sits precisely at the back end-point of the spacecraft and happens to be passing Brian just as Susan passes Brian the second time. Alice and Dave, who had synchronized their watches carefully, determine the time between events E and F by subtracting the time of event E measured by Alice from the time of event F measured by Dave.

a) (1 points) Answer each of the four questions below by writing the appropriate letter: S=Susan, B=Brian, AD=Alice and Dave. If more than one of these choices correctly answers the question, be sure to supply *all* answers that are valid.

- Who measures a proper time between events E and F? **Answer:** S and B (NB a spacetime interval is just the longest possible proper time.)
- Who measures the spacetime interval between these events? **Answer:** B
- Who measures the shortest time between the events? **Answer:** S (NB Susan measures a proper time along a non-inertial worldline. This is always less than the spacetime interval, which is always less than or equal to a coordinate time.)

$$\Delta t \geq \Delta s \geq \Delta \tau \tag{1}$$

- Who measures the longest time between the events? **Answer:** AD. They measure the coordinate time between events E and F with a pair of synchronized clocks in an inertial RF (their spacecraft frame)

b) (1 points) What is the difference between the time that Brian measures and the time that Susan measures between these events? **Please show your work**.

Answer: Susan measures a proper time between E and F, which is given by

$$\Delta \tau = \int_{t_E}^{t_F} dt \sqrt{1 - v(t)^2} \tag{2}$$

In this case the speed is constant $v = 3/5$. The times t_E and t_F can be measured in any inertial RF. In this case they are measured by Brian (and he measures also the spacetime interval): we have $\Delta s = \Delta t_{Brian} = t_F - t_E = 500$ ns. Therefore

$$\Delta \tau = (t_F - t_E)\sqrt{1 - v^2} = 500\sqrt{1 - \left(\frac{3}{5}\right)^2} = 400\,\text{ns} \tag{3}$$

and the difference is $\Delta s - \Delta \tau = 100$ ns.

c) (1 points) Knowing that Brian sits at $x = 500\,\text{ns}$ (in SR units) from the origin of the Home Frame and that the spacecraft of Alice and Dave moves with constant speed $\beta = 2/5$ along the +x direction of the Home Frame, draw a two-observer diagram where the Other Frame is the frame where the spacecraft of Alice and Dave is at rest. In particular, **draw and label** the t and x axes for the Home Frame and the t' and x' axes for the Other Frame in this diagram. **Calibrate** these axes as carefully and accurately as you can.

Answer: The slope of the x' and t' axes w.r.t. the x and t axes, respectively, is $\beta = 2/5$. Calibration is done according to the factor $\gamma = 1/\sqrt{1-\beta^2} = 5/\sqrt{21} \simeq 1.09$. More precisely, every interval $\Delta t'$ on the t' axis will measure $\Delta t = \gamma \Delta t'$ on the t axis, and every interval $\Delta x'$ on the x' axis will measure $\Delta x = \gamma \Delta x'$ on the x axis.

$$\text{For}\Delta t' = 100\,\text{ns} \qquad \Delta t = \frac{5}{\sqrt{21}} 100\,\text{ns} \simeq 1.09 \times 100\,\text{ns} = 109\,\text{ns} \tag{4}$$

We choose, for convenience, the origin of the Other Frame, $x' = 0$ and $t' = 0$ to coincide with the origin of the Home Frame (where Brian sits at $x = 500\,\text{ns}$) $x = 0$ and $t = 0$. The position in space of Brian is not relevant for the construction of the two-observer diagram axes and their calibration. Figure 1 shows the axes of the Home Frame t, x and the Other Frame t', x', both calibrated. Time and spatial distances are measured in ns.

d) (0.5 points) Draw and label the worldlines of Brian, Alice and Dave. Indicate and label events E and F.

Answer: Figure 1 shows the worldlines of Brian, Alice and Dave, together with the events E and F.

e) (1 points) What is the time difference between events E and F measured by Alice and Dave? Use the algebraic method with the appropriate equations and compare your result with the graphical method.

Answer: The coordinate time between E and F measured by Alice and Dave is the coordinate time measured in the Other Frame $\Delta t' = t'_F - t'_E$. Knowing that Brian measures $\Delta t = 500\,\text{ns}$ and $\Delta x = 0$ in the Home Frame, we can use the Lorentz coordinate transformations to go from the Home Frame to the Other Frame

$$\Delta t' = \gamma \Delta t - \gamma \beta \Delta x = \frac{5}{\sqrt{21}} 500\,\text{ns} \simeq 1.09 \times 500\,\text{ns} = 545\,\text{ns} \tag{5}$$

On the two-observer diagram the time coordinates t'_E and t'_F are obtained by drawing lines parallel to the x' axis from events E and F and reading out the t' coordinates of the intersections with the t' axis. The approximate value that we read on the diagram is 550 ns in good agreemt with the algebraic determination.

f) (1 points) What is the length of the spacecraft of Alice and Dave as measured by Alice and Dave? Use the algebraic method with the appropriate equations and compare your result with the graphical method.

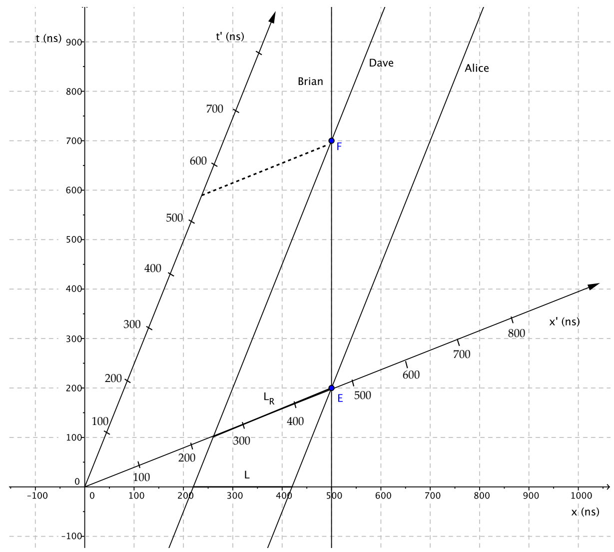

Figure 1:

Answer: Alice and Dave measure the rest length of their spacecraft L_R. This is also given by the absolute value of the spatial coordinate difference $\Delta x'$ between events E and F in the Other Frame. This is obtained by using the Lorentz coordinate transformations from the Home Frame to the Other Frame

$$\Delta x' = \gamma \Delta x - \gamma\beta\Delta t = -\gamma\beta\Delta t = -\frac{5}{\sqrt{21}}\frac{2}{5}500\,\text{ns} \simeq -1.09 \times 200\,\text{ns} = -218\,\text{ns} \tag{6}$$

where we have used that $\Delta x = 0$ for events E and F in the Home Frame. Hence, $L_R = 218\,\text{ns}$.

The length L_R is also indicated on the two-observer diagram. There we read approximately 220ns, in good agreement with the result of the algebraic determination.

g) (0.5 points) What is the length of the spacecraft of Alice and Dave as measured by Brian? Use the algebraic method with the appropriate equations and compare your result with the graphical method.

Answer: We use Lorentz contraction to find that Brian measures $L = \frac{1}{\gamma}L_R = \beta\Delta t = \frac{2}{5} \times 500\,\text{ns} = 200\,\text{ns}$, which is indeed less than L_R.

The length L is also indicated on the two-observer diagram. There we read approximately 200ns, in good agreement with the result of the algebraic determination. Also possible to use a geometrical construction on the diagram, using triangles.

2. (3 points) An object with mass m sits at rest. A light flash moving in the $-x$ direction with a total energy of $\frac{3}{2}m$ hits this object and is completely absorbed. What are the mass and the $x-$velocity of the object afterward?

a) (1 points) Answer this question approximately, using an energy-momentum diagram.

Answer: Figure 2 shows the energy-momentum diagram for this process. The energy and the x-component of the tri-momentum are measured in units of the mass m. We call $\vec{V}_1$ the vector representing the components of the flash of light four-momentum before absorption, $\vec{V}_2$ the vector representing the components of the four-momentum of the object at rest before absorbing light, and $\vec{V}_3$ the vector representing the components of the four-momentum of the object afterward. The final energy E_3 and the x-component of the tri-momentum P_{3x} can

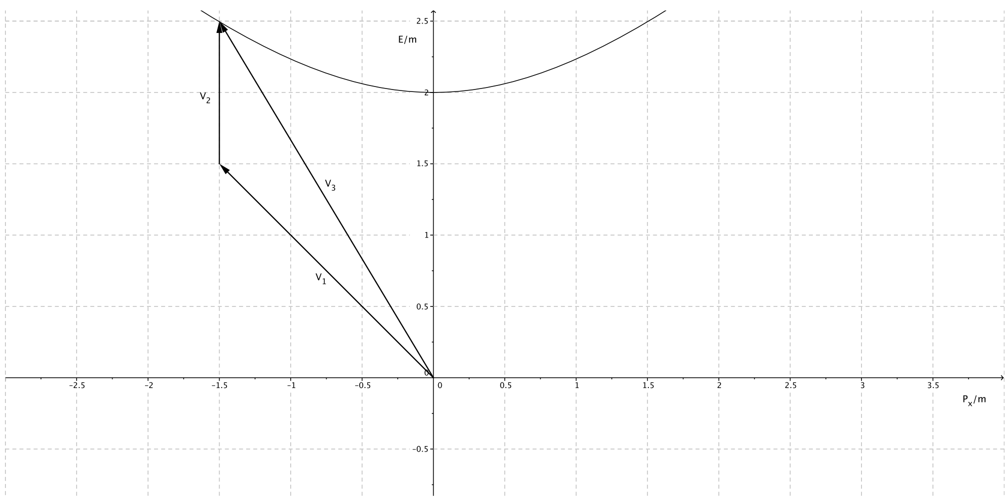

Figure 2:

be read on the diagram by drawing from $\vec{V}_3$ parallels to the P_x and E axis, respectively. We read $E_3/m = 5/2$ and $P_{3x}/m = -3/2$. The final velocity v_{3x} is given by the inverse of the slope of the vector $\vec{V}_3$, i.e. $\frac{1}{v_{3x}} = -\frac{(5/2)}{(3/2)} = -\frac{5}{3}$.

The mass of the final object can be read on the diagram as the intercept with the energy axis of the hyperbola with equation $(E/m)^2 - (P_x/m)^2 = 4$, which passes through $\vec{V}_3$. We thus obtain $M = 2m$.

b) (2 points) Solve this problem algebraically, using four-dimensional column vectors.

Answer: For the flash of light we have $E_1 = -P_{1x} = \frac{3}{2}m$, and $P_{1y,z} = 0$. For the object before absorbing the light $E_2 = m$ and $P_{2x,y,z} = 0$. After the light flash is absorbed, the object has energy E_3 and tri-momentum components $P_{3x,y,z}$. We have to find the mass M of the object in the final state and its $x-$velocity v_{3x}. The conservation of total four-momentum gives

(write column vectors or equations, it does not matter provided it is correctly formulated)

$$
\begin{aligned}
E_1 + E_2 &= E_3 & \frac{3}{2}m + m &= E_3 \\
P_{1x} + P_{2x} &= P_{3x} & -\frac{3}{2}m &= P_{3x} \\
P_{1y} + P_{2y} &= P_{3y} & 0 &= P_{3y} \\
P_{1z} + P_{2z} &= P_{3z} & 0 &= P_{3z}
\end{aligned} \tag{7}
$$

Using that $P_{3y,z} = 0$, $E_3 = \sqrt{M^2 + P_{3x}^2}$, and $v_{3x} = P_{3x}/E_3$, with $E_3 = \frac{5}{2}m$ and $P_{3x} = -\frac{3}{2}m$, one obtains

$$
\begin{aligned}
M &= \sqrt{\frac{25}{4} - \frac{9}{4}} = 2m \\
v_{3x} &= -\frac{3}{5}
\end{aligned} \tag{8}
$$